

Estimation of Acoustic Propagation Uncertainty Through Polynomial Chaos Expansions

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Abstract – *The estimation of sonar system performance in uncertain environments requires the transfer of sound speed, bathymetric and sea surface uncertainty to propagation uncertainty. Currently, methods of estimating the transfer of environmental uncertainty have been based on Monte Carlo simulation of acoustic propagation through an ensemble of environments, or on simplifying assumptions about the physics of propagation either through inhomogeneous media or in the presence of non-separable boundary conditions. In this paper we introduce the Polynomial Chaos expansion method, which allows the integration of the differential equations of acoustic propagation over a basis expansion that is orthogonal with respect to the probability distribution function describing the environmental variability. The resulting solution converges in a relatively small number of expansion coefficients and may be used to estimate any statistical property of the acoustic field, such as its mean value and its variance.*

Keywords: Uncertainty, acoustic propagation, internal waves, Polynomial Chaos.

1 Introduction

The transfer of environmental uncertainty into performance prediction uncertainty in sonar performance modeling requires that the effect of the environmental uncertainty on acoustic propagation be estimated. This is usually undertaken through a modeling exercise where models possessing various degrees of fidelity are exercised over a broad range of environmental variability drawn either from environmental measurements obtained during at-sea experiments or through some objective analysis of environmental databases. In all cases, the goal of the exercise is to increase the robustness of performance prediction for systems that will operate in the presence of the inevitable environmental uncertainty associated with the oceans.

The execution of Monte Carlo runs of acoustic models over environmental ensembles is certainly a viable approach for estimating uncertainty transfer for sonar systems. However, it is usually difficult to learn much about why the acoustic variability occurs, where it occurs

in the field, and why and how the variability deviates from predictions made with simplified models of variability transfer. There is also a relative lack of information about how the statistical distributions of the environmental variability translate into the distributional properties of the acoustic response.

In this paper we adopt an alternative method to describe the transfer of environmental uncertainty to acoustic uncertainty. We adopt the method of Polynomial Chaos expansions, introduced into underwater acoustics by Finette[1], whereby the acoustic field is expanded in a basis which is orthogonal under the probability distribution functions of the environmental variability. As with any appropriate basis expansion of a solution, the daunting task of solving a high dimension problem is reduced to solving a tractable lower dimension problem. In the approach proposed here, the lower dimension problem is also closely analogous to a simplified model of propagation through uncertainty widely used in underwater acoustics: adiabatic normal mode theory. We solve the differential equations for coupled mode propagation in an inhomogeneous environment with a polynomial chaos expansion used not for the complex modal amplitudes but rather for their natural log. As will be seen, this approach is the natural one for capturing the probability distribution of the phase aberrations caused by environmental variability and is useful for capturing the effects of mode coupling as well.

2 Theory

The one-way coupled mode equations for propagation in range-dependent media are [2]

$$\frac{da_n(x)}{dx} + \sum_{m=1}^N a_m(x) \int_{-\infty}^0 \frac{\partial \phi_m(x, z)}{\partial x} \frac{\phi_n(x, z)}{\rho} dz \pm i k_n(x) a_n(x) = 0, \quad (1)$$

where a_n are the range dependent complex modal amplitudes, ϕ_n are the range and depth dependent normal mode shape functions, ρ is the range and depth dependent density profile and k_n are the range dependent modal wavenumbers. In the absence of mode coupling (1) has the well-known adiabatic solution [3]

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$$a_n(x) = a_n(0) \exp\left(\pm i \int_0^x k_n(x') dx'\right). \quad (2)$$

In this paper we restrict our treatment to uncertainty introduced by sound speed defects in the water column, such as would be introduced by internal wave activity [4]. The sound speed defects of the waveguide may be expanded in an empirical orthogonal function (EOF) basis

$$\frac{\delta c}{c}(x, z) = \sum_{l=1}^L \left[\frac{\delta c}{c} \right]_l(z) \eta_l(x). \quad (3)$$

We seek simple expressions of the intermodal coupling matrix $A_{mn} = \int dz \phi_m \phi_n / \rho$ and the deviations of the modal wavenumber k_n that are caused by the individual EOFs. For the decomposition in (3), we can adopt a perturbative treatment for the mode coupling and the aberrations to the horizontal wavenumber caused by each EOF [5]

$$\Delta k_n^l \approx -\frac{\omega^2}{k_n} \left(\int_{-\infty}^0 \frac{\phi_n^2(z)}{c^2(z) \rho(z)} \left[\frac{\delta c}{c} \right]_l(z) dz \right) \quad (4a)$$

$$A_{mn}^l \approx \frac{2\omega^2}{k_m^2 - k_n^2} \left(\int_{-\infty}^0 \frac{1}{r} \frac{\phi_m(z) \phi_n(z)}{c^2(z) \rho(z)} \left[\frac{\delta c}{c} \right]_l(z) dz \right). \quad (4b)$$

Equation (1) may then be written as

$$\frac{d\tilde{a}_n(x)}{dx} + \sum_{m=1}^N \tilde{a}_m A_{mn}^l \frac{d\eta_l(x)}{dx} - i \Delta k_n^l \eta_l(x) \tilde{a}_n(x) = 0 \quad (5)$$

where $\tilde{a}_n = a_n \exp(-ik_n x)$.

2.1 Polynomial Chaos expansion of \tilde{a}_n

The polynomial chaos (PC) basis expansion technique, based on the theory of Cameron and Martin [6], is in its simplest form a way of expanding a solution to a differential equation in a basis orthogonal under a weight function which is the probability density function (*pdf*) of the uncertain environmental parameters. For environmental uncertainties which are distributed according to a Gaussian *pdf*, the appropriate basis are the modified Hermite polynomials Γ [6] which obey the relation

$$\begin{aligned} \langle \Gamma_j(\xi) \Gamma_k(\xi) \rangle &= \int He_j(\xi) He_k(\xi) \exp(-\xi^2/2) d\xi \\ &= k! \delta_{jk} \end{aligned} \quad (6)$$

For clarity the first four modified Hermite polynomials He are shown below. Note that they are explicit functions of the Gaussian distributed random variable ξ

$$\begin{aligned} He_1(\xi) &= \xi \\ He_2(\xi) &= \xi^2 - 1 \\ He_3(\xi) &= \xi^3 - 3\xi \\ He_4(\xi) &= \xi^4 - 6\xi^2 + 3 \end{aligned} \quad (7)$$

In the following we assume that range dependent coefficient η_l of the l^{th} EOF conforms to a Gaussian distribution

$$\eta_l(x) = N(0, \sigma_l^2 \tilde{\eta}_l(x)). \quad (8)$$

We have many choices as to how we expand \tilde{a}_n in a modified Hermite basis. The most obvious expansion

$$\tilde{a}_n = \sum_{j=1}^{N_{PC}} \tilde{b}_{nj}(x) He_j(\xi) \quad (9)$$

is dissatisfying because even for adiabatic propagation the solution would be slowly converging

$$\frac{d\tilde{b}_{ln}}{dx} l! = -i \Delta k_n \sum_{k=1}^{N_{PC}} \eta(x) \langle \xi He_k(\xi) He_l(\xi) \rangle \tilde{b}_{kn}(x). \quad (10)$$

This is because for adiabatic propagation Gaussian distributed sound speed aberrations would cause Gaussian distributed phase aberrations to \tilde{a}_n . The expansion (9) would therefore attempt to build up a log-normal distribution from a product of modified Hermite polynomials whose coefficients were distributed Gaussian; inefficient at best, and very likely slowly converging.

Motivated by the fact that under the adiabatic approximation \tilde{a}_n would be distributed log normal, a more efficient representation is to expand the *natural logarithm* of \tilde{a}_n in a polynomial chaos basis

$$\tilde{a}_n = \tilde{a}_n(0) \exp\left(\sum_{j=1}^{N_{PC}} \Theta_{jn}(x) He_j(\xi)\right). \quad (11)$$

In what follows the utility of this choice seems to be borne out by rapid convergence of the PC expansion.

2.2 Polynomial Chaos solution for \tilde{a}_n

Inserting (11) into (5) we obtain

$$\sum_j^{N_{IC}} \frac{d\Theta_j}{dx} H_{\epsilon_j}(\xi) = i\Delta k_n^l \sigma_l \tilde{\eta}_l(x) \xi \quad (12)$$

$$-\sum_{m=1}^N \frac{a_m(0)}{a_n(0)} e^{ik_{m,n}^x} \exp\left(\sum_k^{N_{IC}} \Theta_{k,n}(x) H_{\epsilon_k}(\xi) - \sum_j^{N_{IC}} \Theta_{j,n}(x) H_{\epsilon_j}(\xi)\right) A_{m,n}' \sigma_l \frac{d\tilde{\eta}_l}{dx} \xi$$

Equation (12) may be solved for small arguments to the exponential on the second line. This « small argument » solution is

$$\frac{d\Theta_j}{dx} = i\Delta k_n^l \sigma_l \tilde{\eta}_l(x) \delta_{j1} - \sum_{m=1}^N \frac{a_m(0)}{a_n(0)} \exp\left(i(k_{m,n}^x - k_n^x)x\right) A_{m,n}' \sigma_l \frac{d\tilde{\eta}_l}{dx} \langle \xi H_{\epsilon_j}(\xi) \rangle \quad (13)$$

$$-\sum_{m=1}^N \frac{a_m(0)}{a_n(0)} \exp\left(i(k_{m,n}^x - k_n^x)x\right) A_{m,n}' \sigma_l \frac{d\tilde{\eta}_l}{dx} \sum_k^{N_{IC}} \Theta_{k,n}(x) - \Theta_{j,n}(x) \langle H_{\epsilon_k}(\xi) H_{\epsilon_j}(\xi) \rangle$$

Notice the form of (13). In the absence of mode coupling the second and third terms on the RHS are zero, and the resulting solution for Θ_1 is simply

$$\Theta_1(x) = i\Delta k_n^l \sigma_l \int_0^x \tilde{\eta}_l(x') dx' \quad (14)$$

Equation (14) yields the desired log normal form for the modal amplitude envelope

$$\tilde{a}_n = \tilde{a}_n(0) \exp\left(i\Delta k_n^l \sigma_l \xi \int_0^x \tilde{\eta}_l(x') dx'\right), \quad (15)$$

where

$$\xi = N(0,1). \quad (16)$$

Note that in the absence of mode coupling the higher order expansion terms Θ_n , $n=2, \dots, \infty$, are all identically zero. In the presence of modal coupling, these higher order terms are expected to have finite amplitude. In addition, all terms are expected to be complex instead of purely imaginary, with the real part representing the effects of amplitude fluctuations. In these cases, (12) must be numerically integrated for a specific form of $\tilde{\eta}_l$ in order to obtain a solution.

3 Example

Consider 500 Hz acoustic propagation through an internal wave field causing sound speed aberrations conforming to the realization shown in figure 1. The EOFs corresponding to this realization are shown in figure 2, and the corresponding energy and correlation length scales are shown in figures 3 and 4.

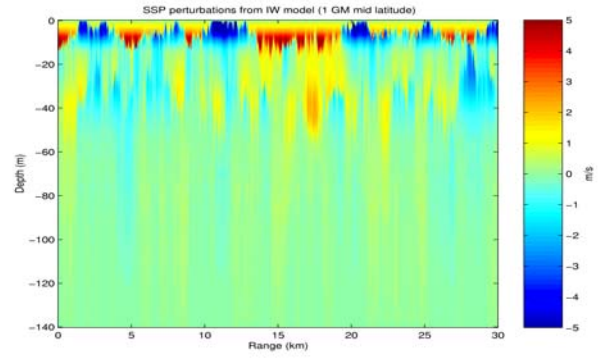


Figure 1 : Sound speed fluctuations caused by internal waves

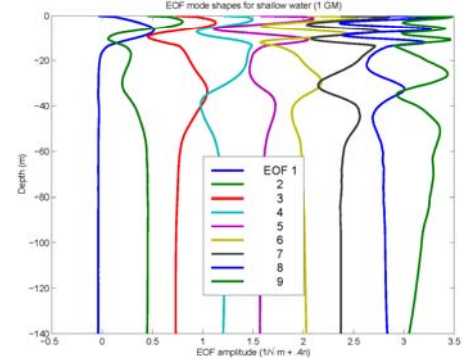


Figure 2 : EOF mode shapes for sound speed defects illustrated in figure 1

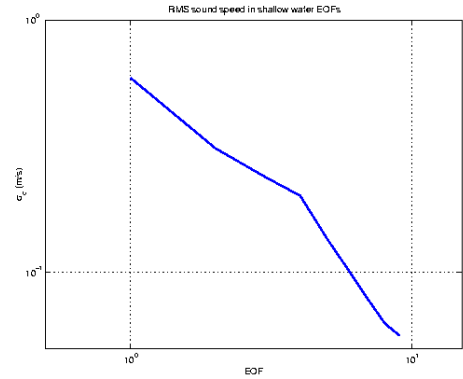


Figure 3 : Energy as a function of EOF

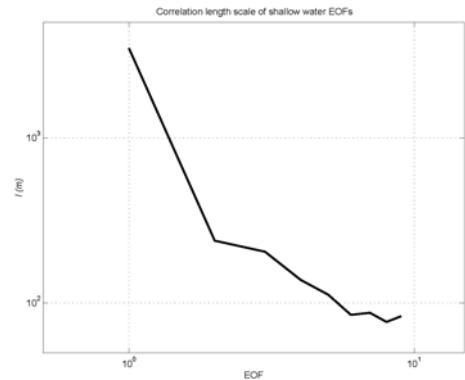


Figure 4 : Horizontal correlation length scale as a function of EOF

Here we evaluate the statistics of the field caused by variability conforming to the most energetic EOF (EOF 1). For the range dependent EOF amplitude $\tilde{\eta}_1$ we choose the form

$$\tilde{\eta}_1 = \sin(r/l_1), \quad (17)$$

where the horizontal correlation length l_1 is set to 3.5 km consistent with figure 4.

Equation (13) is integrated using the Δk_n^1 and A_{mn}^1 derived for the first EOF mode shape shown in figure 2, and using the above definition of $\tilde{\eta}_1$. The PC expansion is truncated at four terms. The results are shown in figures 5 through 9. Figure 5 shows the imaginary part of the PC coefficients as a function of range and mode number. The range and mode dependence of the imaginary part of the first PC coefficient is very similar to the result which would be obtained with the adiabatic approximation, while the second and higher order terms are seen to decay rapidly with PC order, with the fourth expansion term being approximately 5 orders of magnitude less energetic than the first expansion term. Note that the structure of the higher order terms is more closely identified with modal coupling than it is with the nearly adiabatic accumulation of phase demonstrated by the first coefficient.

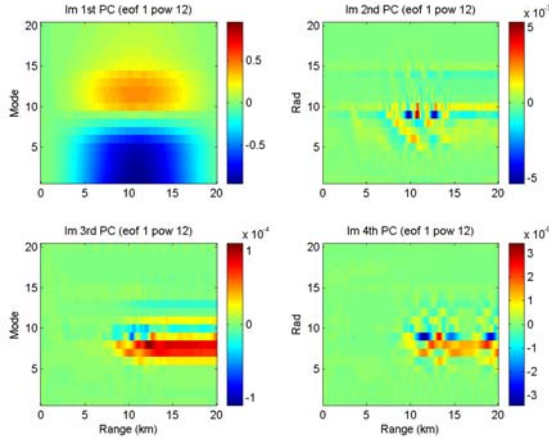


Figure 5 : Imaginary part of first 4 PC coefficients for the most energetic EOF (EOF 1)

Some indication of the non-adiabaticity of the propagation through the first EOF is seen by inspection of the real part of the PC expansion coefficients, shown on a log scale in figure 6. The first expansion coefficient has a non-negligible real amplitude associated with modal coupling, with a maximum magnitude of approximately 10^{-1} . The real part of the higher order expansion coefficients is negligible, the conclusion being that for the first EOF, the propagation statistics of both the real and the imaginary part of the complex modal coefficients are essentially log-normal.

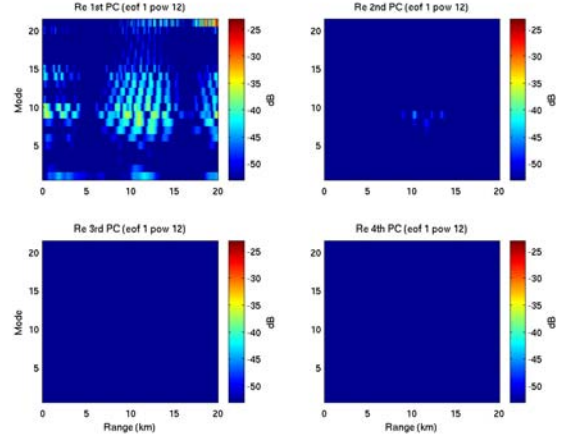


Figure 6 : Real part of first 4 PC coefficients for EOF 1

In figures 7 and 8 we compare the agreement between PC estimates of the phase and log-amplitude standard deviation of propagation through EOF 1 and the Monte Carlo estimates. These figures show that the phase and log-amplitude standard deviation agreement between the two methods is quite good, giving confidence in the correctness of the PC method. Note that the PC results were obtained in a time 20 times longer than individual Monte Carlo calculations, of which 200 were required to obtain the results shown.

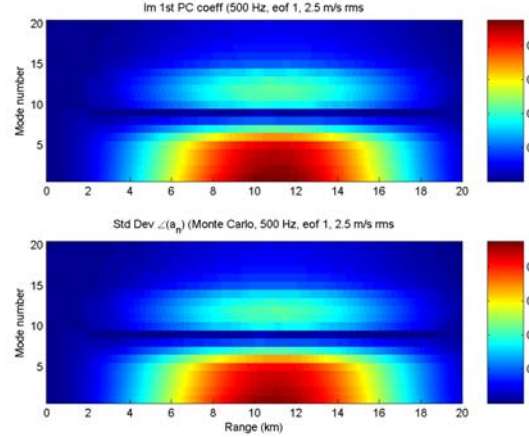


Figure 7: Comparison between PC estimate of EOF 1 phase standard deviation (top) and MC estimate (bottom)

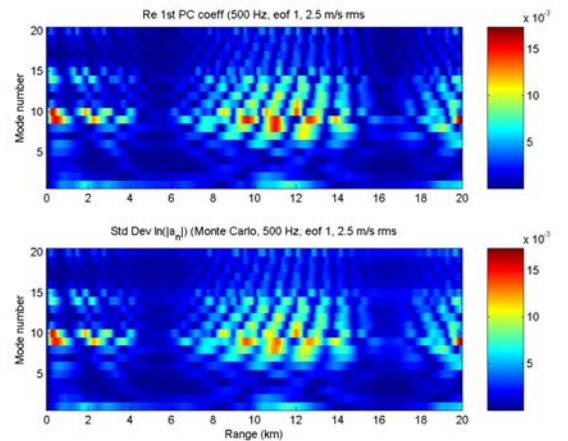


Figure 8 : Comparison between PC estimate of EOF 1 log amplitude standard deviation (top) and MC estimate

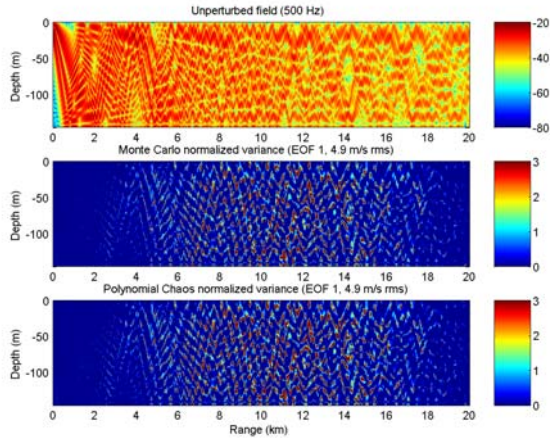


Figure 9 : Comparison between $10\log_{10}$ of the PC estimate of the EOF 1 scintillation index (middle) and the MC estimate (bottom). Top figure unperturbed transmission loss at 500 Hz

Finally in figure 9 $10\log_{10}$ of the second moment of the coherent pressure field intensity in the presence of EOF 1 is plotted normalized by the unperturbed transmission loss (shown in the top panel) in the second panel for the Monte Carlo estimate and in the third panel for the PC expansion. The normalized variance is highest in the deep nulls associated with the unperturbed results: this is consistent with the fact that the deep nulls are associated with perfect destructive interference which is unpredictable in the presence of EOF 1. The largest variances in the PC solution are found at ranges between 4 and 18 km, this is due to the nature of the form of $\tilde{\eta}_1$ chosen in (16), which reaches its maximum argument at ranges of $r = \pi(n+1/2)l$ where as mentioned before $l=3.5$ km.

4 Conclusions

A Polynomial Chaos expansion for the complex modal amplitudes of coupled acoustic normal modes propagating through internal wave activity has been derived and implemented. Results show that expressions for the standard deviation of the modal phase and log-amplitude obtained by integrating the PC equations agree very closely with estimates of these quantities estimated by Monte Carlo methods. The PC expansion chosen was also shown to be very rapidly converging in the number of expansion terms required. The speed of the PC calculations was roughly 10 times faster than the Monte Carlo estimate method, and the number of required expansion terms gave a degree of insight into the importance of the non-linear transfer of uncertainty associated with acoustic propagation through internal waves, since the existence of orders higher than 1 indicates non-log-normal propagation statistics. It is hoped that the Polynomial Chaos method will continue to provide a fruitful avenue of research into the non-linear transfer of environmental uncertainty into acoustic propagation uncertainty, that tools based on this theory can be developed for the rapid and robust estimation of uncertainty parameters required for operation research purposes, and that the method can be expanded to treat

problems of time domain propagation, including scattering and reverberation, in an analogous manner to that outlined here.

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